Chapter 6
Lecture 1

Canonical Transformations

Akhlaq Hussain

6.1 Canonical Transformations

Hamiltonian formulation $H(q_i, p_i) = \sum_{i=1}^{N} p_i \, \dot{q}_i - L$ (Hamiltonian)

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

(*Hamiton's Equations*)

one can get the same differential equations to be solved as are provided by the Lagrangian procedure.

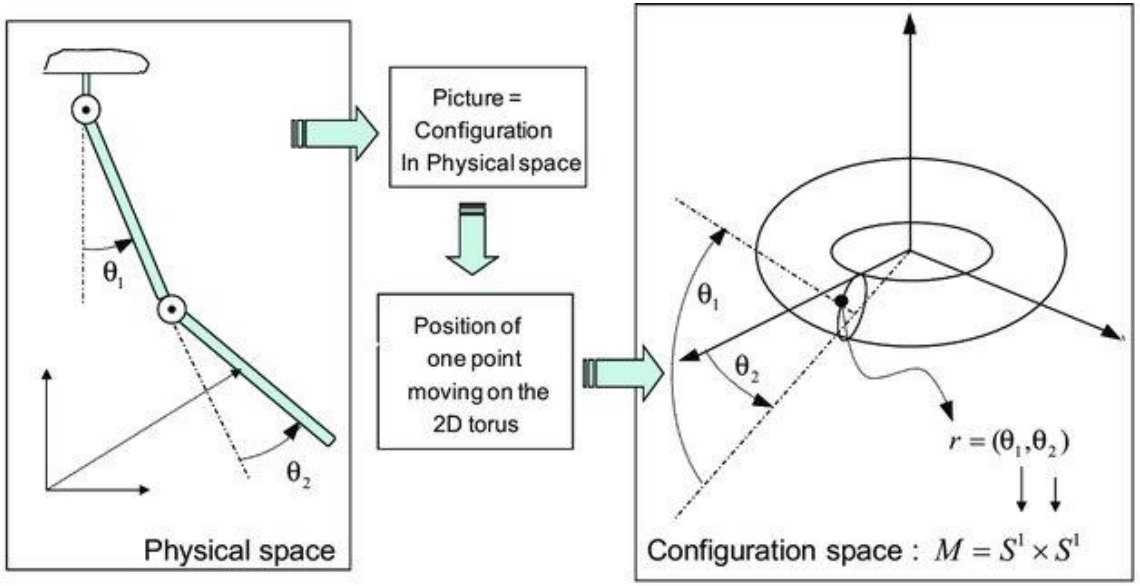
$$L(\dot{q}_i, q_i) = T - V \qquad \text{(Lagrangian)}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad \text{(Lagrange's equation)}$$

Therefore, the Hamiltonian formulation does not decrease the difficulty of solving Problems. The advantages of Hamiltonian formulation is not its use as a calculation tool, but rather in deeper insight it offers into the formal structure of the mechanics.

6.1 Canonical Transformations

- In Lagrangian mechanics $\{L(\dot{q}_i, q_i)\}$ system is described by "q_i" and velocities" $\dot{q}_i$" in configurational space,
- The parameters that define the configuration of a system are called generalized coordinates and the vector space defined by these coordinates is called configuration space.
- The position of a single particle moving in ordinary Euclidean Space (3D) is defined by the vector $q = q(x, y, z)$ and therefore its configuration space is $Q = \mathbb{R}^3$
- For n disconnected, non-interacting particles, the configuration space is $\mathbb{R}^{3n}$.

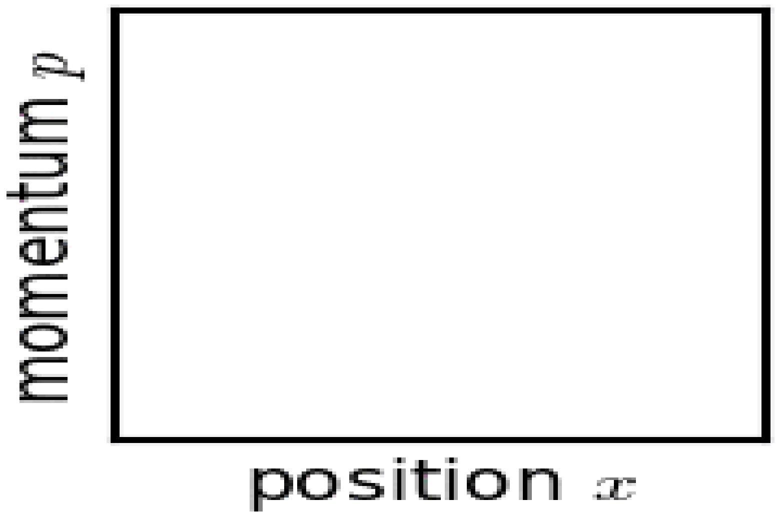


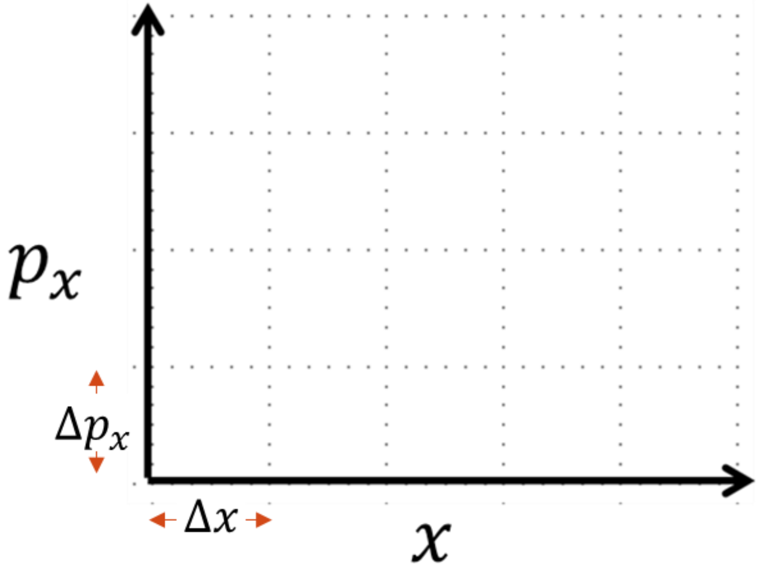
6.1 Canonical Transformations

- In Hamiltonian $\{H(q_i, p_i)\}$ we describe the state of the system in Phase space by generalized coordinates and momenta.

- In dynamical system theory, a Phase space is a space in which all possible states of a system are represented with each possible state corresponding to one unique point in the phase space.

- There exist different momenta for particles with same position and vice versa.





6.1 Canonical Transformations

- To understand the importance of Hamiltonian let us consider a problem for which solution of Hamilton's equations are trivial (simple) and Hamiltonian is constant of motion.
- For this case all the coordinates "q_i" of the problem will be cyclic and all conjugate "p_i" momenta will be constant.

Since $$p_i = \alpha_i = Constant$$

And $$\dot{q}_i = \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial \alpha_i} = \omega_i$$

$$q_i = \omega_i t + \beta_i$$

β_i is constant and can be find by the initial conditions.

But in real problem it is not necessary that all the coordinates are cyclic.

6.1 Canonical Transformations

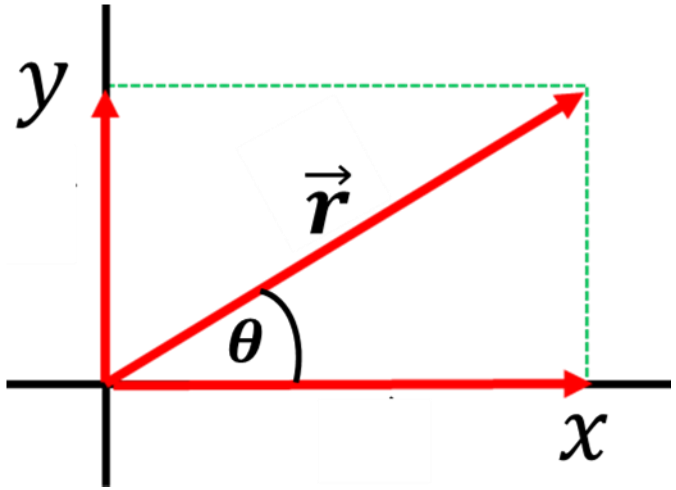
- Practically, it rarely happens that all the coordinates are cyclic.
- However a system can be described by more than one set of generalized coordinates.
- The motion of particle in plane is described by generalized coordinates either the cartesian coordinates.

In cartesian coordinates

$$q_1 = x, \quad \& \quad q_1 = y$$

In polar coordinates

$$q_1 = r, \quad \& \quad q_1 = \theta$$



Both choices are equally valid, but one of the set may be more convenient for the problem under the consideration. Not that for the central force neither x, nor y is cyclic while the second set does contain a cyclic coordinate θ

6.1 Canonical Transformations

- The number of cyclic coordinates thus depend on choice of generalized coordinates, and for each problem there may be one choice for which all the coordinates are cyclic.

- Since the generalized coordinates suggested by the problem will not be cyclic normally, we must first derive a specific procedure for transforming from one set of variables to some other set that may be more suitable.

6.1 Canonical Transformations

- Let us consider transformation equations

$$Q_i = Q_i(q_i, p_i, t), \ \& \ \ P_i = P_i(q_i, p_i, t)$$

- Such that the general dynamical theory is invariant under these transformations.

- Let us consider a function $K(Q_i, P_i, t)$ such that

$$\boxed{\dot{P}_i = -\frac{\partial K}{\partial Q_i}} \quad \& \quad \boxed{\dot{Q}_i = \frac{\partial K}{\partial P_i}}$$

Q_i & P_i are called canonical coordinates and transformation $q_i \longrightarrow Q_i$ & $p_i \longrightarrow P_i$

$Q_i = Q_i(q_i, p_i, t), \ \& \ \ P_i = P_i(q_i, p_i, t)$ are known as canonical transformations.

6.1 Canonical Transformations

Here "$K(Q_i, P_i, t)$" play role of Hamiltonian and Q_i & P_i must satisfy Hamilton's principle.

$$\delta \int_{t_1}^{t_2} \left[\sum P_i \dot{Q}_i - K(Q_i, P_i, t)\right] dt \qquad (1)$$

$$\delta \int_{t_1}^{t_2} \left[\sum p_i \dot{q}_i - H(q_i, p_i, t)\right] dt = 0 \qquad (2)$$

Equation (1) and Equation (2) may not be qual, therefore we can find a function "F" such that

$$\int_{t_1}^{t_2} \frac{dF}{dt} dt = F(t_2) - F(t_1)$$

and $\delta \int_{t_1}^{t_2} \frac{dF}{dt} dt = 0$ where $\delta F(t_2) = \delta F(t_1)$

and

$$\sum p_i \dot{q}_i - H(q_i, p_i, t) = \sum P_i \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF}{dt} \qquad (3)$$

Function "F" is called generating function.

6.1 Canonical Transformations

- There are four different possibilities for "F"

1) $F_1(q_i, Q_i, t)$ Provided that q_i, Q_i are treated as independent

2) $F_2(q_i, P_i, t)$ Provided that q_i, P_i are treated as independent

3) $F_3(p_i, Q_i, t)$ Provided that p_i, Q_i are treated as independent

4) $F_4(p_i, P_i, t)$ Provided that p_i, P_i are treated as independent

6.1 Canonical Transformations

Case I: Eq. 3 can be written as

$$\sum p_i\,\dot{q}_i - H(q_i,p_i,t) = \sum P_i\,\dot{Q}_i - K(Q_i,P_i,t) + \frac{dF_1}{dt} \qquad (4)$$

Since $\frac{dF_1(q_i,Q_i,t)}{dt} = \sum \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t}$ Therefore Eq. (4)…

$$\sum p_i\,\dot{q}_i - H(q_i,p_i,t) = \sum P_i\,\dot{Q}_i - K(Q_i,P_i,t) + \sum \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum \frac{\partial F_1}{dQ_i}\dot{Q}_i + \frac{\partial F_1}{\partial t}$$

Comparing coefficient of "$\dot{q}_i$" & "$\dot{Q}_i$" on both sides

$$p_i = \frac{\partial F_1}{\partial q_i} \qquad (5)a$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} \qquad (5)b$$

And $$K(Q_i,P_i,t) = H(q_i,p_i,t) + \frac{\partial F_1}{\partial t} \qquad (5)c$$

6.1 Canonical Transformations

- From Eq. (5a) we can determine “p_i” in terms of q_i, Q_i and t and the inverse transformation Q_i in terms of q_i, p_i *and* t

Eq. (5)a ⇒ $Q_i = Q_i(q_i, p_i, t)$

Eq. (5)b ⇒ $P_i = P_i(q_i, p_i, t)$

& Eq. (5)c provide connection between new and old Hamiltonian

6.1 Canonical Transformations

Case II: For $F_2(q_i, P_i, t)$ generating function

Since $P_i = -\frac{\partial F_1}{\partial Q_i}$

Therefore, $$F_1(q_i, Q_i, t) + \sum P_i\, Q_i = F_2(q_i, P_i, t)$$

Since $$\sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_1}{dt}$$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{d}{dt}\left[F_2(q_i, P_i, t) - \sum P_i\ \ Q_i\right]$$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_2}{dt} - \sum P_i\, \dot{Q}_i - \sum \dot{P}_i\, Q_i$$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = -K(Q_i, P_i, t) + \frac{dF_2}{dt} - \sum \dot{P}_i\, Q_i$$

Since $$\frac{d}{dt} F_2(q_i, P_i, t) = \sum \frac{\partial F_2}{\partial q_i} \dot{q}_i + \sum \frac{\partial F_2}{\partial P_i} \dot{P}_i + \frac{\partial F_2}{\partial t}$$

6.1 Canonical Transformations

Putting in previous equation

$$\sum p_i \dot{q}_i - H(q_i, p_i, t) = -K(Q_i, P_i, t) + \sum \frac{\partial F_2}{\partial q_i} \dot{q}_i + \sum \frac{\partial F_2}{\partial P_i} \dot{P}_i + \frac{\partial F_2}{\partial t} - \sum \dot{P}_i Q_i$$

Comparing coefficient of "$\dot{q}_i$" & "$\dot{P}_i$" on both sides

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad (6)\text{a}$$

$$Q_i = \frac{\partial F_1}{\partial P_i} \qquad (6)\text{b}$$

And

$$K(Q_i, P_i, t) = H(q_i, p_i, t) + \frac{\partial F_2}{\partial t} \qquad (6)\text{c}$$

6.1 Canonical Transformations

Case III: For $F_3(p_i, Q_i, t)$ generating function

Since $p_i = \frac{\partial F_1}{\partial q_i}$

Therefore, we can write $F_1(q_i, Q_i, t) - \sum p_i\, q_i = F_3(p_i, Q_i, t)$

Since $\sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_1}{dt}$

$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{d}{dt}[F_3(p_i, Q_i, t) + \sum p_i\, q_i]$

$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_3}{dt} + \sum p_i\, \dot{q}_i + \sum \dot{p}_i\, q_i$

$\Rightarrow -H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_3}{dt} + \sum \dot{p}_i\, q_i$

Since $\frac{d}{dt} F_3(p_i, Q_i, t) = \sum \frac{\partial F_3}{\partial p_i} \dot{p}_i + \sum \frac{\partial F_3}{\partial Q_i} \dot{Q}_i + \frac{\partial F_3}{\partial t}$, putting in above equation.

6.1 Canonical Transformations

Putting in previous equation

$$-H(q_i, p_i, t) = \sum P_i \dot{Q}_i - K(Q_i, P_i, t) + \sum \frac{\partial F_3}{\partial p_i} \dot{p}_i + \sum \frac{\partial F_3}{\partial Q_i} \dot{Q}_i + \frac{\partial F_3}{\partial t} + \sum \dot{p}_i q_i$$

Comparing coefficient of "$\dot{p}_i$" & "$\dot{Q}_i$" on both sides

$$q_i = -\frac{\partial F_3}{\partial p_i} \qquad (7)\text{a}$$

$$P_i = -\frac{\partial F_3}{\partial Q_i} \qquad (7)\text{b}$$

And

$$K(Q_i, P_i, t) = H(q_i, p_i, t) + \frac{\partial F_3}{\partial t} \qquad (7)\text{c}$$

6.1 Canonical Transformations

Case IV: For $F_4(p_i, P_i, t)$ generating function

Since $p_i = \frac{\partial F_1}{\partial q_i}$ & $P_i = -\frac{\partial F_1}{\partial Q_i}$

Therefore, we can write $F_1(q_i, Q_i, t) - \sum p_i\, q_i + \sum P_i\, Q_i = F_4(p_i, Q_i, t)$

Since $\sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_1}{dt}$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{d}{dt}\left[F_4(p_i, Q_i, t) + \sum p_i\, q_i - \sum P_i\, Q_i\right]$$

$$\Rightarrow \sum p_i\, \dot{q}_i - H(q_i, p_i, t) = \sum P_i\, \dot{Q}_i - K(Q_i, P_i, t) + \frac{dF_4}{dt} + \sum p_i\, \dot{q}_i + \sum \dot{p}_i\, q_i - \sum P_i\, \dot{Q}_i - \sum \dot{P}_i\, Q_i$$

$$-H(q_i, p_i, t) = -K(Q_i, P_i, t) + \frac{dF_4}{dt} + \sum \dot{p}_i\, q_i - \sum \dot{P}_i\, Q_i$$

Since $\frac{d}{dt} F_4(p_i, P_i, t) = \sum \frac{\partial F_4}{\partial p_i} \dot{p}_i + \sum \frac{\partial F_4}{\partial P_i} \dot{P}_i + \frac{\partial F_4}{\partial t}$, putting in above equation.

6.1 Canonical Transformations

Putting in previous equation

$$-H(q_i, p_i, t) = -K(Q_i, P_i, t) + \sum \frac{\partial F_4}{\partial p_i}\dot{p}_i + \sum \frac{\partial F_4}{\partial P_i}\dot{P}_i + \frac{\partial F_4}{\partial t} + \sum \dot{p}_i\, q_i - \sum \dot{P}_i\, Q_i$$

Comparing coefficient of "$\dot{p}_i$" & "$\dot{P}_i$" on both sides

$$q_i = -\frac{\partial F_4}{\partial p_i} \qquad (8)a$$

$$Q_i = \frac{\partial F_4}{\partial P_i} \qquad (8)b$$

And $$K(Q_i, P_i, t) = H(q_i, p_i, t) + \frac{\partial F_4}{\partial t} \qquad (8)c$$

6.1 Canonical Transformations

Properties of the Four basic canonical transformations

Generating function	Derivatives of generating function	Trivial special cases	Transformation
$F_1(q_i, Q_i, t)$	$p_i = \frac{\partial F_1}{\partial q_i}, P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$	$p_i = Q_i,$ $P_i = -q_i$
$F_2(q_i, P_i, t)$	$p_i = \frac{\partial F_2}{\partial q_i}, Q_i = \frac{\partial F_1}{\partial P_i}$	$F_2 = q_i P_i$	$p_i = P_i$ $Q_i = q_i$
$F_3(p_i, Q_i, t)$	$q_i = -\frac{\partial F_3}{\partial p_i}, P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$	$q_i = -Q_i$ $P_i = -p_i$
$F_4(p_i, P_i, t)$	$q_i = -\frac{\partial F_4}{\partial p_i}, Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$	$q_i = -P_i$ $Q_i = p_i$

6.2 Conditions for the transformation to be canonical

Conditions for the transformation to be canonical

For $F_1(q_i, Q_i, t) \Rightarrow \quad dF_1 = \sum p_i dq_i - \sum P_i dQ_i$

For $F_2(q_i, P_i, t) \Rightarrow \quad dF_2 = \sum p_i dq_i + \sum Q_i dP_i$

For $F_3(p_i, Q_i, t) \Rightarrow \quad dF_3 = -\sum q_i dp_i - \sum P_i dQ_i$

For $F_4(p_i, P_i, t) \Rightarrow \quad dF_4 = -\sum q_i dp_i + \sum P_i dQ_i$

6.2 Conditions for the transformation to be canonical

The transformation from (q_i, p_i) to (Q_i, P_i) will be canonical if

$$\sum \boldsymbol{p_i dq_i} - \sum \boldsymbol{P_i dQ_i}$$

is an exact differential

Solution: Consider the generating function $F_1(q_i, Q_i)$

$$dF_1 = \sum \frac{\partial F_1}{\partial q_i} dq_i + \sum \frac{\partial F_1}{\partial Q_i} dQ_i$$

Since $\quad p_i = \frac{\partial F_1}{\partial q_i} \quad$ and $\quad P_i = -\frac{\partial F_1}{\partial Q_i}$

Therefore,

$$dF_1 = \sum p_i dq_i - \sum P_i dQ_i$$

which is an exact differential equation.

6.2 Conditions for the transformation to be canonical

Similarly, considering generating function $F_4(p_i, P_i, t)$

$$dF_4(p_i, P_i, t) = \sum \frac{\partial F_4}{\partial p_i} dp_i + \sum \frac{\partial F_4}{\partial P_i} dP_i$$

Since $q_i = -\frac{\partial F_4}{\partial p_i}$ and $Q_i = \frac{\partial F_4}{\partial P_i}$

Therefore, $dF_4(p_i, P_i, t) = -\sum q_i dp_i + \sum Q_i dP_i$ which is an exact differential

Now subtracting dF_4 from dF_1

$$dF_1 - dF_4 = \sum p_i dq_i - \sum P_i dQ_i + \sum q_i dp_i - \sum Q_i dP_i$$

$$\Rightarrow dF_1 - dF_4 = \left(\sum q_i dp_i + \sum p_i dq_i\right) - \left(\sum Q_i dP_i + \sum P_i dQ_i\right)$$

$$\Rightarrow dF_1 - dF_4 = d(q_i p_i) - d(Q_i P_i)$$

$$\Rightarrow d(F_1 - F_4) = d(q_i p_i - Q_i P_i)$$

Which is exact differential. Therefore the transformation is canonical.

And $\Rightarrow F_1 = F_4 + q_i p_i - Q_i P_i$